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Constructions of new orthogonal arrays and covering arrays of strength three ☆

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ABSTRACT

A covering array of size N, strength t, degree k, and order v, or a CA$(N;t,k,v)$ in short, is a $k \times N$ array on v symbols. In every $t \times N$ subarray, each t-tuple column vector occurs at least once. When 'at least' is replaced by 'exactly', this defines an orthogonal array, OA(t,k,v). A difference covering array, or a DCA$(k,n;v)$, over an abelian group G of order v is a $k \times n$ array (a_{ij}) $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n)$ with entries from G, such that, for any two distinct rows l and h of D $(1 \leqslant l < h \leqslant k)$, the difference list $\Delta_{lh} = \{d_{h1} - d_{l1}, d_{h2} - d_{l2}, \ldots, d_{hn} - d_{ln}\}$ contains every element of G at least once.

Covering arrays have important applications in statistics and computer science, as well as in drug screening. In this paper, we present two constructive methods to obtain orthogonal arrays and covering arrays of strength 3 by using DCAs. As a consequence, it is proved that there are an OA$(3,5,v)$ for any integer $v \geqslant 4$ and $v \not\equiv 2 \pmod 4$, and an OA$(3,6,v)$ for any positive integer v satisfying $\gcd(v,4) \neq 2$ and $\gcd(v,18) \neq 3$. It is also proved that the size CAN$(3,k,v)$ of a CA$(N;3,k,v)$ cannot exceed $v^3 + v^2$ when $k = 5$ and $v \equiv 2 \pmod 4$, or $k = 6$, $v \equiv 2 \pmod 4$ and $\gcd(v,18) \neq 3$.

1. Introduction

A *covering array*, or a CA$(N;t,k,v)$ in short, is a $k \times N$ array with entries from a set of v symbols. In every $t \times N$ subarray, each t-tuple column vector occurs at least once. Then t is the *strength* of the coverage of interactions, k is the number of components (*degree*), and v is the number of symbols for

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each component (*order*). The size N is omitted when inessential in the context. The minimum size N for which a $\mathrm{CA}(N;t,k,v)$ exists is called a *covering array number* and written as $\mathrm{CAN}(t,k,v)$.

When the requirement 'at least' is replaced by 'exactly' in the definition of a $\mathrm{CA}(N;t,k,v)$, the defined object is known as an orthogonal array in the literature and denoted by $\mathrm{OA}(N;t,k,v)$, or $\mathrm{OA}(t,k,v)$ briefly. This means that an $\mathrm{OA}(N;t,k,v)$ is just a $\mathrm{CA}(N;t,k,v)$ with $N=v^t$. An $\mathrm{OA}(N;2,k,v)$ is equivalent to a set of $k-2$ mutually orthogonal Latin squares of side v in the case $t=2$. When $t\geqslant 3$, it is also equivalent to a transversal t-design.

As is well known (see [1,7,22]), covering arrays, especially orthogonal arrays, are of importance in design theory. They have also various applications in statistics, coding theory and computer science, as well as in drug screening. On this aspect, the interested reader may refer to [2,6,13,21,22]. The determination of the function $\mathrm{CAN}(t,k,v)$ has been the subject of much research (see, for example, [5,7,11,12]). Covering arrays were also studied as t-surjective arrays by Seroussi and Bshouty [18], transversal coverings by Stevens and Mendelsohn [19,20] and t-qualitatively independent partitions of a set of size N by Körner et al. [15,17]. The following remarkable result on the existence of an $\mathrm{OA}(t,k,v)$ of $t\geqslant 3$ is due to Bush [4].

Theorem 1.1. *(See [4].) If n is a prime power and $t<n$, then an $\mathrm{OA}(t,n+1,n)$ exists. Moreover, if $n\geqslant 4$ is a power of 2, an $\mathrm{OA}(3,n+2,n)$ exists.*

In this paper, we focus our attention on covering arrays of strength 3. We present two constructive methods to obtain orthogonal arrays and covering arrays of strength 3. As a consequence, it is proved that there are an $\mathrm{OA}(3,5,v)$ for any integer $v\geqslant 4$ and $v\not\equiv 2 \pmod 4$, and an $\mathrm{OA}(3,6,v)$ for any positive integer v satisfying $\gcd(v,4)\neq 2$ and $\gcd(v,18)\neq 3$. It is also proved that the size $\mathrm{CAN}(3,k,v)$ of a $\mathrm{CA}(N;3,k,v)$ cannot exceed v^3+v^2 when $k=5$ and $v\equiv 2 \pmod 4$, or $k=6$, $v\equiv 2 \pmod 4$ and $\gcd(v,18)\neq 3$.

2. The first construction

Our first constructive method uses difference covering arrays (DCAs).

Let G be an abelian group of order v. Following [24], a difference covering array, or a $\mathrm{DCA}(k,n;v)$ is a $k\times n$ array (a_{ij}) $(1\leqslant i\leqslant k,\ 1\leqslant j\leqslant n)$ with entries from G, such that, for any two distinct rows l and h of D $(1\leqslant l<h\leqslant k)$, the difference list

$$\Delta_{lh}=\{d_{h1}-d_{l1},d_{h2}-d_{l2},\ldots,d_{hn}-d_{ln}\}$$

contains every element of G at least once. When 'at least' is replaced by 'exactly', this defines a difference matrix ($(v,k;1)$-DM). A DM (DCA) over a cyclic group of order v is said to be cyclic and denoted by CDM (CDCA).

Difference covering arrays and difference matrices have been proved to be very useful in the construction of covering arrays. It was shown in [7,24] that a $\mathrm{DCA}(k,n;v)$ can be used to construct a $\mathrm{CA}(vn;2,k,v)$. The detailed information regarding difference matrices, the reader may refer to [1,8] and the references therein.

Zhu and the first author [14] used a $(g,4;1)$-DM to construct g pairwise disjoint transversal designs $\mathrm{TD}(2,4,g)$, which is equivalent to a $\mathrm{TD}(3,5,g)$ [16]. Our first construction stated in the following theorem can be regarded as a generalization of the above ideas.

Theorem 2.1. *If there exists a $\mathrm{DCA}(4,n;v)$, then there exists a $\mathrm{CA}(v^2n;3,5,v)$.*

Proof. Let $D=(d_{ij})$ be the given $\mathrm{DCA}(4,n;v)$ over the abelian group G. For each column

$$(d_{1j},d_{2j},d_{3j},d_{4j})^T$$

of the DCA, construct the following columns:

$$C(j,u,e)=(d_{1j}+u,d_{2j}+u,d_{3j}+u+e,d_{4j}+u+e,e)^T,$$

where $e,u\in G$.

Table 1

v	Old bound	New bound
10	1219 [5]	1100
12	1991 [5]	1728
14	3107 [5]	2940
18	6443 [5]	6156
21	10100 [5]	9261
22	12165 [8]	11132
24	14927 [5]	13824

It is left to check the matrix M consisting of the above $v^2 \times n$ columns is a CA$(3, 5, v)$. We need only to show that any column $B = (x, y, z)^T$ occurs on rows a, b, c of M at least once, where $1 \leqslant a < b < c \leqslant 5$.

Case 1. Suppose that $c = 5$. If $(a, b) \in \{(1, 2), (3, 4)\}$, there is an integer $j \in I_n = \{1, 2, \dots, n\}$ such that $d_{bj} - d_{aj} = y - x$ since D is a DCA$(4, n; v)$ over G. Then B occurs in the column $C(j, x - d_{1j}, z)$ on rows a, b, c of M when $(a, b) = (1, 2)$, and in the column $C(j, x - z - d_{3j}, z)$ when $(a, b) = (3, 4)$. If $(a, b) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$, there is an integer $j \in I_n$ such that $d_{bj} - d_{aj} = y - z - x$. Then B occurs in the column $C(j, x - d_{aj}, z)$ on rows a, b, c of M.

Case 2. Suppose that $c < 5$. If $(a, b, c) \in \{(1, 2, 3), (1, 2, 4)\}$, there is an integer $j \in I_n$ such that $d_{bj} - d_{aj} = y - x$. Then B occurs in the column $C(j, x - d_{aj}, z - x - d_{cj} + d_{aj})$ on rows a, b, c of M. If $(a, b, c) \in \{(1, 3, 4), (2, 3, 4)\}$, there is an integer $j \in I_n$ such that $d_{cj} - d_{bj} = z - y$. Then B occurs in the column $C(j, x - d_{aj}, z - x - d_{cj} + d_{aj})$ on rows a, b, c of M. The proof is complete. $\square$

If we start with a $(v, 4; 1)$-DM, instead of a DCA$(4, n; v)$, then we obtain the following known construction from Theorem 2.1.

It was proved [9,23] that a $(v, 4; 1)$-DM can exist only if $v \not\equiv 2 \pmod 4$, while a $(v, 4; 1)$-CDM can exist only if v is odd.

Corollary 2.2. *(See [14,16].) If there exists a* $(v, 4; 1)$-DM, *then there exists an* OA$(3, 5, v)$.

Lemma 2.3. *(See [10].) Let $v \geqslant 4$ be an integer. If $v \not\equiv 2 \pmod 4$, then a* $(v, 4; 1)$-DM *exists.*

Lemma 2.4. *(See [23].) For all even positive integers v, there exists a* CDCA$(4, v + 1; v)$.

From Lemma 2.3 and Lemma 2.4, we can apply Corollary 2.2 and Theorem 2.1 to establish the following results.

Theorem 2.5. *Let $v \geqslant 4$ be an integer. If $v \not\equiv 2 \pmod 4$, then an* OA$(3, 5, v)$ *exists.*

Theorem 2.6. *For any even positive integers v, there exists a* CA$(v^3 + v^2; 3, 5, v)$, *and hence* CAN$(3, 5, v) \leqslant v^3 + v^2$.

The results established in Theorems 2.5 and 2.6 lower the upper bounds for those known CAN$(3, 5, v)$ which we display in Table 1.

3. The second construction

3.1. The description of the construction

Our second construction uses difference covering arrays satisfying one more requirement.

Let $D=(d_{ij})$ be a DCA$(4,n;v)$ over an abelian group G. An n-tuple $s=(s_1,s_2,\dots,s_n)$ over G is called an *adder* of the difference covering array D if each element of G appears in the multiset $\{s_1,s_2,\dots,s_n\}$ at least once and the matrix

$$D^s=\left(d'_{ij}\right),\quad \text{where } d'_{ij}=d_{ij} \text{ for } i\in\{1,2\} \text{ and } d'_{ij}=d_{ij}+s_j \text{ for } i\in\{3,4\},$$

is also a DCA$(4,n;v)$ over the group G.

Lemma 3.1. *There is a* CDCA$(4,3;2)$ *and a* CDCA$(4,11;10)$ *associated with an adder.*

Proof. A CDCA$(4,3;2)$ over Z_2 is given by the following array:

$$\left(\begin{array}{ccc} 0&0&0\\ 0&1&1\\ 0&0&1\\ 0&1&0\\ \hline 0&1&1 \end{array}\right).$$

Here, the first four rows of the array form a DCA$(4,3;2)$ and the last row is its corresponding adder.

A CDCA$(4,11;10)$ over Z_{10} is given by the following array:

$$\left(\begin{array}{ccccccccccc} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&2&3&4&5&6&7&8&9\\ 0&1&3&6&8&2&0&9&4&7&5\\ 0&3&7&4&1&9&9&5&8&2&6\\ \hline 0&6&1&2&4&3&7&2&5&9&8 \end{array}\right).$$

As above, the first four rows form a CDCA$(4,11;10)$ and the last row is its corresponding adder. $\square$

Making use of a DCA$(4,n;v)$ associated with an adder, we can present our second construction.

Theorem 3.2. *If there is a* DCA$(4,n;v)$ *associated with an adder, then there is a* CA$(v^2n;3,6,v)$.

Proof. Let $D=(d_{ij})$ and $s=(s_1,s_2,\dots,s_n)$ be the given DCA$(4,n;v)$ over the group G and its corresponding adder, respectively.

For each column $(d_{1j},d_{2j},d_{3j},d_{4j})^T$ of the DCA, construct the following columns:

$$C(j,u,e)=(d_{1j}+u,d_{2j}+u,d_{3j}+u+e+s_j,d_{4j}+u+e+s_j,e,e+s_j)^T,$$

where $e,u\in G$.

It is left to check the matrix M consisting of the above $v^2\times n$ columns is a CA$(3,6,v)$.

Consider the submatrix consisting of rows $1,2,3,4$ and 6. Similar to the proof of Theorem 2.1, it is a CA$(3,5,v)$ since D is a DCA$(4,n;v)$. Similarly, the submatrix consisting of rows 1,2,3,4 and 5 is also a CA$(3,5,v)$ since D^s is also a DCA$(4,n;v)$ by the definition of an associated adder. Then we need only to show that for $i\in\{1,2,3,4\}$ each submatrix consisting of rows $i,5,6$ contains each column vector $B=(x,y,z)^T$ at least once.

Since each group element appears in the associated adder s at least once, there is an integer $j\in I_n$ such that $s_j=z-y$. If $i\in\{1,2\}$, then B occurs in the column $C(j,x-d_{ij},y)$ on rows $i,5,6$ of M. If $i\in\{3,4\}$, then B occurs in the column $C(j,x-z-d_{ij},y)$ on rows $i,5,6$ of M. The proof is complete. $\square$

As an immediate consequence, we have the following useful corollary.

Corollary 3.3. *If there exists a* $(v,4;1)$-DM *associated with an adder, then there exists an* OA$(3,6,v)$.

3.2. *New* OA$(3,6,v)$

In order to obtain new OA$(3,6,v)$s by applying Corollary 3.3, we need $(v,4;1)$-DMs associated with an adder. As noted earlier, a $(v,4;1)$-DM cannot exist when $v \equiv 2 \pmod 4$. In this subsection, we construct such DMs in the case where $\gcd(v,18) \neq 3$.

Lemma 3.4. *For any prime power $q \geqslant 4$, there is a $(q,4;1)$-DM associated with an adder.*

Proof. Let G be the additive group of the finite field GF(q) of order $q \geqslant 4$. Let ξ be an arbitrary primitive element of GF(q). Consider the following $(q,4;1)$-DM over G:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \xi^0 & \xi^1 & \xi^2 & \cdots & \xi^{q-2} \\ 0 & \xi^1 & \xi^2 & \xi^3 & \cdots & \xi^0 \\ 0 & \xi^2 & \xi^3 & \xi^4 & \cdots & \xi^1 \end{pmatrix}.$$

If we take $s = (0, \xi^0, \xi^1, \xi^2, \dots, \xi^{q-2})$, then we have the matrix

$$D^s = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \xi^0 & \xi^1 & \xi^2 & \cdots & \xi^{q-2} \\ 0 & \xi^1+\xi^0 & \xi^2+\xi^1 & \xi^3+\xi^2 & \cdots & \xi^0+\xi^{q-2} \\ 0 & \xi^2+\xi^0 & \xi^3+\xi^1 & \xi^4+\xi^2 & \cdots & \xi^1+\xi^{q-2} \end{pmatrix}.$$

It is easy to see that both D and D^s are a $(q,4;1)$-DM over G. Hence, the D forms a $(q,4;1)$-DM associated with the adder s. $\square$

Lemma 3.5. *There is a* $(12,4;1)$-DM *associated with an adder.*

Proof. Take G as the additive group of $Z_6 \times Z_2$, write each element $(x,y) \in G$ as xy. Then the following array gives us a $(12,4;1)$-DM associated with an adder:

$$\left(\begin{array}{cccccccccccc} 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 & 00 \\ 00 & 01 & 10 & 11 & 20 & 21 & 30 & 31 & 40 & 41 & 50 & 51 \\ 00 & 10 & 01 & 40 & 30 & 51 & 20 & 50 & 41 & 21 & 31 & 11 \\ 00 & 11 & 40 & 20 & 41 & 01 & 50 & 21 & 31 & 10 & 51 & 30 \\ \hline 00 & 21 & 10 & 11 & 40 & 41 & 30 & 51 & 20 & 01 & 50 & 31 \end{array}\right).$$

Remark that the first four rows of the above array form a $(12,4;1)$-DM and the last row is the corresponding adder. $\square$

Lemma 3.6. *There is a* $(24,4;1)$-DM *associated with an adder.*

Proof. Consider the following matrix over $Z_3 \times Z_2 \times Z_2 \times Z_2$, where each element $(x,y,z,w) \in Z_3 \times Z_2 \times Z_2 \times Z_2$ is written as $xyzw$,

$$\left(\begin{array}{cccccccccccc} 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 & 0000 \\ 0000 & 0010 & 0100 & 0110 & 1000 & 1010 & 1100 & 1110 & 2000 & 2010 & 2100 & 2110 \\ 0000 & 0011 & 1001 & 2110 & 0111 & 2011 & 2111 & 1000 & 0100 & 1100 & 1101 & 2010 \\ 0000 & 1010 & 1011 & 2000 & 1101 & 2110 & 0001 & 0101 & 2100 & 2001 & 0111 & 1100 \\ \hline 0000 & 0010 & 1100 & 2010 & 2001 & 2101 & 0011 & 2110 & 1111 & 0001 & 1110 & 1101 \end{array}\right).$$

For the desired DM, we replace each column $(a,b,c,d,e)^T$ by $(a,b,c,d,e)^T$ and $(a+(0,0,0,0), b+(0,0,0,1), c+(0,0,1,0), d+(0,0,1,1), e+(0,1,1,0))^T$. Then the first four rows of the resulting matrix form a $(24,4;1)$-DM over $Z_3 \times Z_2 \times Z_2 \times Z_2$ [8] and the last row is the corresponding adder. $\square$

Lemma 3.7. *(See [3].) If* $\mathrm{OA}(t,k,v_i)$ *for* $1 \leqslant i \leqslant m$ *all exist, then so does an* $\mathrm{OA}(t,k,\prod_{i=1}^m v_i)$.

Now we can establish new orthogonal arrays of strength 3.

Theorem 3.8. *Let* v *be a positive integer which satisfies* $\gcd(v,4) \neq 2$ *and* $\gcd(v,18) \neq 3$. *Then there is an* $\mathrm{OA}(3,6,v)$.

Proof. The result for $v \in \{12, 24\}$ follows from applying Corollary 3.3, in conjunction with Lemma 3.5 and Lemma 3.6. For other values of v, we write $v = 2^\alpha 3^\beta p_1^{\gamma_1} p_2^{\gamma_1} \cdots p_r^{\gamma_r}$ for its prime factorization, where $p_j \geqslant 5$. By assumption, we know that $\alpha \neq 1$ and $(\alpha, \beta) \neq (0, 1)$. If $\beta \neq 1$, by Lemma 3.7 there is an $\mathrm{OA}(3,6,v)$ since there are an $\mathrm{OA}(3,6,p_j^{\gamma_j})$, an $\mathrm{OA}(3,6,3^\beta)$ and an $\mathrm{OA}(3,6,2^\alpha)$ from Theorem 1.1. If $\beta = 1$, then $\alpha \geqslant 2$. When α is even, by Lemma 3.7 there is an $\mathrm{OA}(3,6,v)$ since both an $\mathrm{OA}(3,6,12)$ and an $\mathrm{OA}(3,6,2^{\alpha-2} p_1^{\gamma_1} p_2^{\gamma_1} \cdots p_r^{\gamma_r})$ exist. When α is odd, by Lemma 3.7 there is an $\mathrm{OA}(3,6,v)$ since both an $\mathrm{OA}(3,6,24)$ and an $\mathrm{OA}(3,6,2^{\alpha-3} p_1^{\gamma_1} p_2^{\gamma_1} \cdots p_r^{\gamma_r})$ exist. □

3.3. New upper bounds for $\mathrm{CAN}(3,6,v)$

In this subsection, we apply Theorem 3.2 to improve the known upper bounds of $\mathrm{CAN}(3,6,v)$. We will use the notion of an holey difference matrix used in the construction of difference packing (covering) arrays in [23,24].

Let G be an abelian group of order v which contains a subgroup H of order w. A $k \times (v-w)$ matrix $D = (d_{ij})$ $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant v-w)$ with entries from G is said to be a holey difference matrix (HDM) with one hole if, for any two distinct rows l and h of D $(1 \leqslant l < h \leqslant k)$, the difference list

$$\Delta_{lh} = \{d_{h1} - d_{l1}, d_{h2} - d_{l2}, \dots, d_{h(v-w)} - d_{l(v-w)}\}$$

contains every element of $G \setminus H$ exactly once, while any element of H does not appear in Δ_{lh} (and hence H is a hole). For convenience, we shall refer to such a matrix D as a $(k,v;w)$-HDM over $(G;H)$. A $(k,v;w)$-HDM gives us $k-2$ holey mutually orthogonal latin squares of side v (see [8]).

Let $D = (d_{ij})$ be a $(4,v;w)$-HDM over $(G;H)$. A $(v-w)$-tuple $s = (s_1, s_2, \dots, s_{v-w})$ over $(G;H)$ is called an adder of the holey difference matrix D if all s_i are in $G \setminus H$ and pairwise distinct, and the matrix

$$D^s = \left(d'_{ij}\right), \quad \text{where } d'_{ij} = d_{ij} \text{ for } i \in \{1,2\} \text{ and } d'_{ij} = d_{ij} + s_j \text{ for } i \in \{3,4\}$$

is also a $(4,v;w)$-HDM over $(G;H)$.

The following two working lemmas are simple, but useful.

Lemma 3.9. *Suppose that there is a* $(4,q;w)$-HDM *associated with an adder over* $(G;H)$. *If there is a* $\mathrm{DCA}(4,n;w)$ *associated with an adder over* H, *then there is a* $\mathrm{DCA}(4,q-w+n;q)$ *associated with an adder over* G.

Proof. Let A be a $(4,q;w)$-HDM associated with an adder s^A over $(G;H)$. Let B be a $\mathrm{DCA}(4,n;w)$ associated with an adder s^B over H. Then $D = (A \mid B)$ is a $\mathrm{DCA}(4,q-w+n;q)$ associated with the adder $s = (s^A \mid s^B)$ over G. □

Lemma 3.10. *Suppose that there is a* $(4,q;w)$-HDM *associated with an adder over* $(G;H)$. *If there is a* $(4,w;w_1)$-HDM *associated with an adder over* $(H;H_1)$, *then there is a* $(4,q;w_1)$-HDM *associated with an adder over* $(G;H_1)$.

Proof. Let A be a $(4,q;w)$-HDM associated with an adder over $(G;H)$. Let B be a $(4,w;w_1)$-HDM associated with an adder s^B over $(H;H_1)$. Then $D = (A \mid B)$ forms a $(4,q;w_1)$-HDM associated with the adder $s = (s^A \mid s^B)$ over $(G;H_1)$. □

Based on Lemma 3.1 and Lemma 3.9 we can obtain a DCA$(4, v+1; v)$ associated with an adder from a $(4, 2m; 2)$-HDM associated with an adder. Let us now turn to the construction of such HDMs.

Lemma 3.11. *There is a* $(4, 2p; 2)$-HDM *associated with an adder for any prime* $p \geqslant 7$.

Proof. Let $Z_p^{\square}$ and $Z_p^{\not\square}$ denote the sets of all quadratic residues and quadratic non-residues modulo p, respectively. Take a quadratic non-residue x modulo p such that $x \neq \pm(1/2)$ and $2x+2 \in Z_p^{\not\square}$. This can be done. For example, we may take x as

$$x = \begin{cases} 1/y, & \text{if } \{2,3\} \subset Z_p^{\square}, \text{ where } y \text{ is the first quadratic non-residue;} \\ 1/3, & \text{if } 2 \in Z_p^{\square} \text{ and } 3 \in Z_p^{\not\square}; \\ 2, & \text{if } 2 \in Z_p^{\not\square} \text{ and } 3 \in Z_p^{\square}; \\ 3, & \text{if } \{2,3\} \subset Z_p^{\not\square}. \end{cases}$$

Let A stand for the following matrix over $(Z_p \times Z_2)$:

$$\begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) \\ (x,0) & (x^2,0) & (x^3,1) & (x^2,1) \\ (2x,0) & (2x^3,1) & (2x^4,0) & (2x^2,1) \\ (3x,0) & (x^2+2x^3,1) & (x^3+2x^4,1) & (3x^2,0) \end{pmatrix}.$$

It is readily calculated that the difference lists $\triangle_{lh}$ $(1 \leqslant l < h \leqslant 4)$ from the above matrix A are

$$\triangle_{12} = \triangle_{34} = \{(x,0), (x^2,0), (x^3,1), (x^2,1)\},$$
$$\triangle_{13} = \triangle_{24} = \{(2x,0), (2x^4,0), (2x^3,1), (2x^2,1)\},$$
$$\triangle_{14} = \{(3x,0), (3x^2,0), (x^2+2x^3,1), (x^3+2x^4,1)\},$$
$$\triangle_{23} = \{(x,0), (x^2,0), (x^2(2x-1),1), (x^3(2x-1),1)\}.$$

From elementary number theory we can see that for any two distinct rows l and h $(1 \leqslant l < h \leqslant 4)$, the difference list $\triangle_{lh}$ is of the form

$$\bigcup_{j \in Z_2} \{(\alpha_{lh}(j), j), (\beta_{lh}(j), j)\}$$

in which $\alpha_{lh}(j)\beta_{lh}(j)$ is a quadratic non-residue modulo p, for any $j \in Z_2$. With this fact, we see that the following array D is a $(4, 2p; 2)$-HDM over $(Z_p \times Z_2 : \{0\} \times Z_2)$:

$$D = (A_0 \mid A_1 \mid \cdots \mid A_{\frac{p-3}{2}})$$

where $A_k = (w^{2k}, 1) \cdot A$ for $k = 0, 1, \ldots, \frac{p-3}{2}$ and w is a primitive root of Z_p.

For the corresponding adder of A, we set $E = ((x,0), (x^2,0), (x^3,1), (x^2,1))$ and consider the matrix

$$B = \begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) \\ (x,0) & (x^2,0) & (x^3,1) & (x^2,1) \\ (2x,0)+(x,0) & (2x^3,1)+(x^2,0) & (2x^4,0)+(x^3,1) & (2x^2,1)+(x^2,1) \\ (3x,0)+(x,0) & (x^2+2x^3,1)+(x^2,0) & (x^3+2x^4,1)+(x^3,1) & (3x^2,0)+(x^2,1) \end{pmatrix}.$$

As with the matrix A, the difference list $\triangle_{lh}$ $(1 \leqslant l < h \leqslant 4)$ of B is of the form

$$\bigcup_{j \in Z_2} \{(\alpha_{lh}(j), j), (\beta_{lh}(j), j)\}$$

in which $\alpha_{lh}(j)\beta_{lh}(j)$ is also a quadratic non-residue modulo p, for any $j \in Z_2$. It follows that the following matrix D^s is also a $(4, 2p; 2)$-HDM over $(Z_p \times Z_2 : \{0\} \times Z_2)$:

$$D^s = (\, B_0 \;\; | \;\; B_1 \;\; | \;\; \cdots \;\; | \;\; B_{\frac{p-3}{2}} \,)$$

where $B_k = (w^{2k}, 1) \cdot B$.

It now turns out that the matrix D is a $(4, 2p; 2)$-HDM associated with the adder

$$s = (\, E_0 \;\; | \;\; E_1 \;\; | \;\; \cdots \;\; | \;\; E_{\frac{p-3}{2}} \,)$$

where $E_k = (w^{2k}, 1) \cdot E$ for $k = 0, 1, \ldots, (p-3)/2$. $\square$

Lemma 3.12. *There is a* $(4, 2q; 2)$-HDM *associated with an adder over* $(\mathrm{GF}(q) \times Z_2 : \{0\} \times Z_2)$ *for* $q \in \{9, 25, 27, 125\}$.

Proof. For each stated prime power q, let ξ be the primitive element of GF(q) with minimal polynomial $f(x)$, where

$$f(x) = \begin{cases} x^2 + x + 2 & \text{if } q = 9; \\ x^2 + x + 2 & \text{if } q = 25; \\ x^3 + 2x^2 + x + 1 & \text{if } q = 27; \\ x^3 + 3x + 2 & \text{if } q = 125. \end{cases}$$

We write S for the unique multiplicative subgroup of order 2 in GF(q), which consists of all non-zero squares of GF(q). Then, we construct a 4×4 matrix A over $\mathrm{GF}(q) \times Z_2$ given by

$$\begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) \\ (1,0) & (\xi,0) & (\xi^2,1) & (\xi,1) \\ (\xi,0) & (\xi^3,1) & (\xi^4,0) & (\xi^2,1) \\ (1+\xi,0) & (\xi+\xi^3,1) & (\xi^2+\xi^4,1) & (\xi+\xi^2,0) \end{pmatrix}.$$

For any two distinct rows l and h $(1 \leqslant l < h \leqslant 4)$ of A, the difference list Δ_{lh} is of the form $(L_{lh}(0) \times \{0\}) \cup (L_{lh}(1) \times \{1\})$ where both $L_{lh}(0)$ and $L_{lh}(1)$ form a complete system of representatives of the multiplicative cosets of S in GF(q). This property of A guarantees that the following array D is a $(4, 2q; 2)$-HDM over $(\mathrm{GF}(q) \times Z_2 : \{0\} \times Z_2)$ where $A_k = (\xi^{2k}, 1) \cdot A$ for $k = 0, 1, \ldots, \frac{q-3}{2}$,

$$D = (\, A_0 \;\; | \;\; A_1 \;\; | \;\; \cdots \;\; | \;\; A_{\frac{q-3}{2}} \,).$$

Next, we take a quadruple $E = ((1,0), (\xi,0), (\xi^2,1), (\xi,1))$ and consider the matrix

$$B = \begin{pmatrix} (0,0) & (0,0) & (0,0) & (0,0) \\ (1,0) & (\xi,0) & (\xi^2,1) & (\xi,1) \\ (\xi,0)+(1,0) & (\xi^3,1)+(\xi,0) & (\xi^4,0)+(\xi^2,1) & (\xi^2,1)+(\xi,1) \\ (1+\xi,0)+(1,0) & (\xi+\xi^3,1)+(\xi,0) & (\xi^2+\xi^4,1)+(\xi^2,1) & (\xi+\xi^2,0)+(\xi,1) \end{pmatrix}.$$

Simple calculations show that

$$\xi(2+\xi)(2+\xi^2) = \begin{cases} \xi \cdot \xi^6 \cdot \xi^5 & \text{if } q = 9; \\ \xi \cdot \xi^{14} \cdot \xi^{13} & \text{if } q = 25; \\ \xi \cdot \xi^4 \cdot \xi^{23} & \text{if } q = 27; \\ \xi \cdot \xi^{119} \cdot \xi^{56} & \text{if } q = 125. \end{cases}$$

This means that $\xi(2+\xi)(2+\xi^2) \in S$ for each q. So, the difference list Δ_{lh} $(1 \leqslant l < h \leqslant 4)$ has the same property as that of A. With this fact, we see that the matrix

$$D^s = (\, B_0 \;\; | \;\; B_1 \;\; | \;\; \cdots \;\; | \;\; B_{\frac{q-3}{2}} \,)$$

is also a $(4, 2q; 2)$-HDM over $(\mathrm{GF}(q) \times Z_2 : \{0\} \times Z_2)$, where $B_k = (\xi^{2k}, 1) \cdot B$ for $k = 0, 1, \ldots, (q-3)/2$. Therefore, the matrix D constructed above is a $(4, 2q; 2)$-HDM over $(\mathrm{GF}(q) \times Z_2 : \{0\} \times Z_2)$ associated with the adder

$$s = (\, E_0 \quad | \quad E_1 \quad | \quad \cdots \quad | \quad E_{\frac{q-3}{2}} \,)$$

where $E_k = (\xi^{2k}, 1) \cdot E$ for $k = 0, 1, \ldots, (q-3)/2$. □

The following lemma is taken from [24].

Lemma 3.13. *Let G and $\bar{G}$ be two abelian groups of orders q and $\bar{q}$, respectively. Suppose that there exists a $(q, k; 1)$-DM over G and a $(k, \bar{q}; u)$-HDM over $(\bar{G}; H)$. Then there exists a $(k, q\bar{q}; qu)$-HDM over $(G \times \bar{G}; G \times H)$.*

As a variation of Lemma 3.13, we have the following lemma.

Lemma 3.14. *Let G and $\bar{G}$ be two abelian groups of orders q and $\bar{q}$, respectively. Suppose that there exists a $(q, 4; 1)$-DM associated with an adder over G and a $(4, \bar{q}; w)$-HDM associated with an adder over $(\bar{G}; H)$. Then there exists a $(4, q\bar{q}; qw)$-HDM associated with an adder over $(G \times \bar{G}; G \times H)$.*

Proof. By assumption, let $A = (a_{ij})$ $(1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant q)$ be a $(q, 4; 1)$-DM associated with an adder $s^A = (s_1^A, s_2^A, \ldots, s_q^A)$. Let $B = (b_{ik})$ $(1 \leqslant i \leqslant 4, 1 \leqslant k \leqslant n = \bar{q} - w)$ associated with an adder $s^B = (s_1^B, s_2^B, \ldots, s_n^B)$.

Form an array and an qn-tuple

$$D = (\, D_1 \quad | \quad D_2 \quad | \quad \cdots \quad | \quad D_q \,),$$
$$s = (\, s_1 \quad | \quad s_2 \quad | \quad \cdots \quad | \quad s_q \,)$$

over $G \times \bar{G}$, where

$$D_j = \begin{pmatrix} (a_{1j}, b_{11}) & (a_{1j}, b_{12}) & \cdots & (a_{1j}, b_{1n}) \\ (a_{2j}, b_{21}) & (a_{2j}, b_{22}) & \cdots & (a_{2j}, b_{2n}) \\ (a_{3j}, b_{31}) & (a_{3j}, b_{32}) & \cdots & (a_{3j}, b_{3n}) \\ (a_{4j}, b_{41}) & (a_{4j}, b_{42}) & \cdots & (a_{4j}, b_{4n}) \end{pmatrix},$$
$$s_j = \big((s_j^A, s_1^B) \quad (s_j^A, s_2^B) \quad \cdots \quad (s_j^A, s_n^B) \big)$$

for $1 \leqslant j \leqslant q$. From Lemma 3.13, D is a $(4, q\bar{q}; qw)$-HDM over $(G \times \bar{G}; G \times H)$. We claim that s is its corresponding adder. In fact, by the definition of an associated adder, s^A and s^B are a permutation of the elements of G and $\bar{G} \setminus H$, respectively. Hence, s is a permutation of the elements of $G \times \bar{G} \setminus (G \times H)$. Further, by definition, the matrix

$$A^{s^A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ a_{31} + s_1^A & a_{32} + s_2^A & \cdots & a_{3q} + s_q^A \\ a_{41} + s_1^A & a_{42} + s_2^A & \cdots & a_{4q} + s_q^A \end{pmatrix}$$

is a $(q, 4; 1)$-DM over G, while the matrix

$$B^{s^B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & a_{2n} \\ b_{31} + s_1^B & b_{32} + s_2^B & \cdots & b_{3n} + s_n^B \\ b_{41} + s_1^B & b_{42} + s_2^B & \cdots & b_{4n} + s_n^B \end{pmatrix}$$

is a $(4,\bar{q};w)$-HDM over $(\bar{G}:H)$. Hence, from Lemma 3.13 the matrix

$$D^s = (\,D_1^s \;\;|\;\; D_2^s \;\;|\;\; \cdots \;\;|\;\; D_q^s\,)$$

is a $(4,q\bar{q};qw)$-HDM over $(G\times\bar{G};G\times H)$. Here, for $1\leqslant j\leqslant q$,

$$D_j^s = \begin{pmatrix} (a_{1j},b_{11}) & (a_{1j},b_{12}) & \cdots & (a_{1j},b_{1n}) \\ (a_{2j},b_{21}) & (a_{2j},b_{22}) & \cdots & (a_{2j},b_{2n}) \\ (a_{3j}+s_j^A, b_{31}+s_1^B) & (a_{3j}+s_j^A, b_{32}+s_2^B) & \cdots & (a_{3j}+s_j^A, b_{3n}+s_n^B) \\ (a_{4j}+s_j^A, b_{41}+s_1^B) & (a_{4j}+s_j^A, b_{42}+s_2^B) & \cdots & (a_{4j}+s_j^A, b_{4n}+s_n^B) \end{pmatrix}.$$

It follows that s is an adder of the HDM, D. The proof is then complete. □

Lemma 3.15. *Let m be a product of the form $3^\alpha 5^\beta p_1^{\gamma_1} p_2^{\gamma_2}\cdots p_r^{\gamma_r}$ with $\alpha\neq 1$ and $\beta\neq 1$. Then there is a $(4,2m;2)$-HDM associated with an adder.*

Proof. We first show that there is a $(4,2q;2)$-HDM associated with an adder for any odd prime power $q\geqslant 7$.

If $q=3^\alpha$, then $\alpha\geqslant 2$. We give the proof by induction. When $\alpha\in\{2,3\}$, a $(4,2q;2)$-HDM associated with an adder over $(\mathrm{GF}(q)\times Z_2;\{0\}\times Z_2)$ exists by Lemma 3.12. Now assume that there is a $(4,2\cdot 3^k;2)$-HDM associated with an adder over $(\mathrm{GF}(3^k)\times Z_2;\{0\}\times Z_2)$ for $2\leqslant k<\alpha$. Since there is a $(9,4;1)$-DM associated with an adder over GF(9) by Lemma 3.4, applying Lemma 3.14 with the $(4,2\cdot 3^{\alpha-2};2)$-HDM associated with an adder over $(\mathrm{GF}(3^{\alpha-2})\times Z_2;\{0\}\times Z_2)$ gives an $(4,2\cdot 3^\alpha;18)$-HDM associated with an adder over $(\mathrm{GF}(3^{\alpha-2})\times\mathrm{GF}(9)\times Z_2:\{0\}\times\mathrm{GF}(9)\times Z_2)$. Applying Lemma 3.10 with the resulting HDM and a $(4,18;2)$-HDM associated with an adder over $(\{0\}\times\mathrm{GF}(9)\times Z_2;\{0\}\times\{0\}\times Z_2)$ gives a $(4,2\cdot 3^\alpha;2)$-HDM associated with an adder over $(\mathrm{GF}(3^{\alpha-2})\times\mathrm{GF}(9)\times Z_2;\{0\}\times\{0\}\times Z_2)$, which is isomorphic to $(\mathrm{GF}(3^\alpha)\times Z_2;\{0\}\times Z_2)$. It follows that there is a $(4,2q;2)$-HDM associated with an adder over $(\mathrm{GF}(q)\times Z_2;\{0\}\times Z_2)$. If $q=5^\beta$ or $q=p_j^{\gamma_j}$, then doing the same procedure as above yields a $(4,2q;2)$-HDM associated with an adder over $(\mathrm{GF}(q)\times Z_2;\{0\}\times Z_2)$.

Doing the same procedure, we easily obtain a $(4,2m;2)$-HDM associated with an adder over $(\mathrm{GF}(3^\alpha)\times\mathrm{GF}(5^\beta)\times\mathrm{GF}(p_1^{\gamma_1})\times\cdots\times\mathrm{GF}(p_r^{\gamma_r})\times Z_2;\{(0,0,\ldots,0)\}\times Z_2)$. □

Lemma 3.16. *For any integer $v\equiv 2 \pmod 4$ and $\gcd(v,18)\neq 3$, there exists a DCA$(4,v+1;v)$ associated with an adder.*

Proof. Write v as a product of the form $2\cdot 3^\alpha 5^\beta p_1^{\gamma_1} p_2^{\gamma_2}\cdots p_r^{\gamma_r}$, where $\alpha\neq 1$ and $p_j\geqslant 7$ $(j=1,2,\ldots,r)$.

If $\beta\neq 1$, then there is a $(4,v;2)$-HDM associated with an adder over $(\mathrm{GF}(3^\alpha)\times\mathrm{GF}(5^\beta)\times\mathrm{GF}(p_1^{\gamma_1})\times\cdots\times\mathrm{GF}(p_r^{\gamma_r})\times Z_2;\{(0,0,\ldots,0)\}\times Z_2)$ by Lemma 3.15. Applying Lemma 3.9 with a DCA$(4,3;2)$ associated with an adder over $\{(0,0,\ldots,0)\}\times Z_2$ in Lemma 3.1 gives a DCA$(4,v+1;v)$ associated with an adder.

If $\beta=1$, then we start with a $(5,4;1)$-DM associated with an adder over GF(5) by Lemma 3.4. Similar to the proof of Lemma 3.15, apply Lemma 3.14 with the known $(4,v/5;2)$-HDM associated with an adder over $(\mathrm{GF}(3^\alpha)\times\mathrm{GF}(p_1^{\gamma_1})\times\cdots\times\mathrm{GF}(p_r^{\gamma_r})\times Z_2;\{(0,0,\ldots,0)\}\times Z_2)$ from Lemma 3.15. We then obtain a $(4,v;10)$-HDM associated with an adder over $(\mathrm{GF}(3^\alpha)\times\mathrm{GF}(p_1^{\gamma_1})\times\cdots\times\mathrm{GF}(p_r^{\gamma_r})\times\mathrm{GF}(5)\times Z_2;\{(0,0,\ldots,0)\}\times\mathrm{GF}(5)\times Z_2)$. Further applying Lemma 3.9 with the known DCA$(4,11;10)$ over $(0,0,\ldots,0)\times\mathrm{GF}(5)\times Z_2$ in Lemma 3.1 gives the result. □

Applying Theorem 3.2 and Lemma 3.16 we obtain a new bound on CAN$(3,6,v)$ as follows.

Theorem 3.17. *For any integer $v\equiv 2 \pmod 4$ and $\gcd(v,18)\neq 3$, CAN$(3,6,v)\leqslant v^3+v^2$.*

Table 2

v	Old bound	New bound
10		1100
12	2112 [5]	1728
14	3289 [5]	2940
18	6749 [5]	6156
22		11132
24	15479 [5]	13824

Before finishing this section, we remark that our result shown in Theorem 3.17 updates known upper bounds on $\mathrm{CAN}(3, 6, v)$ for some values of v. To see this, we make a comparison in Table 2, where the superscripts denoted the sources of the old bounds.

4. An open problem

As already mentioned earlier, there is no $(v, 4; 1)$-DM for $v \equiv 2 \pmod 4$. So, our constructive method to an $\mathrm{OA}(3, 6, v)$ presented in the previous section work only for $v \equiv 0, 1, 3 \pmod 4$. By Theorem 3.8 and Lemma 3.14, to determine the existence spectrum of an $\mathrm{OA}(3, 6, v)$ with $v \equiv 0, 1, 3 \pmod 4$, we need only to treat the case $v = 3p$ with $p > 3$ a prime. For $p = 5, 7$, we do found a $(3p, 4; 1)$-DM associated with an adder, which we state in the following lemma.

Lemma 4.1. *If $p = 5$ or 7, then there exists a $(3p, 4; 1)$-DM associated with an adder.*

Proof. For $p = 5$, the desired $(15, 4; 1)$-DM over Z_{15}, D, and the corresponding adder s are

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 0 & 2 & 7 & 1 & 11 & 4 & 10 & 13 & 3 & 6 & 12 & 14 & 5 & 9 & 8 \\ 0 & 10 & 9 & 8 & 1 & 7 & 4 & 2 & 14 & 5 & 13 & 12 & 11 & 6 & 3 \end{pmatrix},$$

$$s = (0 \quad 9 \quad 1 \quad 4 \quad 2 \quad 14 \quad 7 \quad 12 \quad 6 \quad 8 \quad 10 \quad 5 \quad 11 \quad 3 \quad 13).$$

For $p = 7$, we take the adder s to be $(s_1|s_2)$ where $s_2 = -s_1$ and

$$s_1 = (5 \quad 11 \quad 13 \quad 14 \quad 3 \quad 2 \quad 6 \quad 9 \quad 20 \quad 4).$$

Then the required $(21, 4; 1)$-DM over Z_{21} can be obtained by appending a column of zeroes to $(A| - A)$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 6 & 12 & 14 & 11 & 18 & 1 & 13 & 16 \\ 3 & 16 & 13 & 17 & 1 & 7 & 12 & 11 & 15 & 19 \end{pmatrix}.$$

Applying Corollary 3.3 to the above two DMs produces the following two new OAs. □

Theorem 4.2. *There is an* $\mathrm{OA}(3, 6, 15)$ *and an* $\mathrm{OA}(3, 6, 21)$.

We conclude this paper with the following conjecture.

Conjecture. *There exists a $(3p, 4; 1)$-DM associated with an adder for any prime $p \geqslant 11$, and hence an* $\mathrm{OA}(3, 6, 3p)$.

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